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Best-Response Dynamics in Directed Network Games

By

Péter Bayer[1], György Kozics[2] and Nóra Gabriella Szőke[3]

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[1]Grenoble Applied Economics Lab, Université Grenoble-Alpes, Grenoble

[2]Department of Economics and Business, Central European University, Budapest, kozics_gyorgy@phd.ceu.edu

[3]Institut Fourier, Université Grenoble-Alpes, Grenoble

Abstract

We study public goods games played on networks with possibly non-reciprocal relationships between players. Examples for this type of interactions include one-sided relationships, mutual but unequal relationships, and parasitism. It is well known that many simple learning processes converge to a Nash equilibrium if interactions are reciprocal, but this is not true in general for directed networks. However, by a simple tool of rescaling the strategy space, we generalize the convergence result for a class of directed networks and show that it is characterized by transitive weight matrices. Additionally, we show convergence in a second class of networks; those rescalable into networks with weak externalities. We characterize the latter class by the spectral properties of the absolute value of the network's weight matrix and show that it includes all directed acyclic networks.

Keywords: Networks, externalities, local public goods, potential games.

JEL classification: C72, D62, D85.

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1 Introduction

All social and economic networks feature relationships which cannot be described as a partnership of equals. There are relationships between pairs of agents in which only one party is interested and the other is not. Some relationships are beneficial for one party but harmful for the other. Even when both parties benefit or both are harmed, the extent by which they are affected by their counterpart's decisions is not necessarily equal. Nevertheless, most scientific works, both theoretical and applied, are using models in which the reciprocity of interactions is a fundamental property. These frameworks, simple graphs and weighted networks, have received a lot of recent interest from economic theorists due to them providing a highly accurate, rich, and efficient description of real-life interactions. However for reasons relating to either convenience or convention, non-reciprocal relations, and their relevance to economic theory, are relatively unexplored in network literature.

In particular, most models featuring externalities in networks such as Ballester et al. (2006) and Bramoullé and Kranton (2007) assume reciprocal interactions. These highly influential theoretical papers opened the way for a number of applications, such as R&D expenditure between interlinked firms (König et al., 2019), peer effects (Blume et al., 2010), defense expenditures (Sandler and Hartley, 1995, 2007), and crime (Ballester et al., 2010). Most of the applied literature continues to assume reciprocity of relations and performs equilibrium analysis. However, as it will be apparent from our results, the basis behind using the Nash equilibrium as a prediction in networks with non-reciprocal interactions is much weaker than with only reciprocal ones. Thus, predictions made by models using simple graphs or weighted networks are only justified for applications where the underlying interaction network is shown to consist only of reciprocal relations.

In this paper we extend the theoretical literature of learning in networks to include non-reciprocal relations by the use of directed networks. By doing so we intend to fill a gap in the theory literature and provide a jumping off point for a stronger connection between subsequent applied literature and real-life interaction networks. Our setting generalizes games played on weighted networks: instead of one weight, each link is defined by two distinct weights, one for each direction. The three important types of non-reciprocal relations we highlight are (1) one-directional links with one of the weights of the link being zero and the other being non-zero, e.g., an upstream city along a river affects a neighboring downstream city by polluting the river but not vice versa, (2) parasitic links with one weight being positive and the other being negative, e.g., a criminal organization engaging in the extortion of a small business gains benefits from the interaction but it comes with losses for the business, and (3) mutual but unequal benefit or harm, with both weights being positive or both being negative but they are not equal, e.g., a monopolistic seller and a competitive buyer both benefit from their economic relationship but the gains from the interaction tend to be greater for the seller than for the buyer.

One of the focuses of theoretical literature of public goods and networks is the convergence of learning processes to the Nash equilibrium. This is to provide a behavioral and an evolutionary motivation to use equilibrium as a prediction in applied settings. Under symmetric weight matrices a number of powerful convergence results are known: Stability of Nash equilibria with respect to the continuous best-response dynamic has been established by Bramoullé et al. (2014). Convergence of the continuous best-response dynamic to some Nash equilibrium in all such games has been shown by Bervoets and Faure (2019). Bayer et al. (2019b) shows convergence of a class of one-sided learning processes. Bervoets et al. (2018) constructs a convergent learning process not requiring the sophistication of the best response. Bayer et al. (2019a) studies the impact of a farsighted agent on the process in a population of myopic players. Together, these results allow for an interpretation of the game's Nash equilibria as the results of a sequence of improvements made separately by the players.

All of the above papers assume reciprocal network interactions. Thus, they make use of more general results of games of reciprocal interactions (Dubey et al., 2006; Kukushkin, 2005), as well as that of generalized aggregative games (Jensen, 2010). The latter structure allows for the use of the theory of potential games (Monderer and Shapley, 1996), specifically, best-response potential games (Voorneveld, 2000). The main intuition behind the existence of a differentiable best-response potential function for reciprocal interaction networks is that the potential's Hessian, a symmetric matrix, must correspond to the Jacobian of the system of best-responses, which, in this case, is equal to the network's weight matrix. In this paper, however, we show that the best-response potential structure can be exploited in important classes of networks even when the relations, as expressed by the interaction matrix, are not reciprocal.

We consider convergence to Nash equilibria under one-sided improvement dynamics taking place in discrete time. Starting from a profile of production decisions, in every time period one player receives an opportunity to change her production, while every other player remains on the previous period's level. In the next period, another player receives a revision opportunity, and so on. We consider three versions of the best-response dynamic. In increasing order of generality, these are the pure best-response dynamic, in which every revision takes the player to her current best choice given the actions of others, the best-response-approaching dynamic, in which every revision moves the player into the interval between her current action and her current best choice, and the best-response-centered dynamic, in which every revision reduces the distance between her action and her current best choice. Pure best-response dynamics as described above are widely studied. In directed network games, best-response-approaching dynamics include the naive learning dynamics introduced by Bervoets et al. (2018), while best-response-centered dynamics are studied in Bayer et al. (2019b) in weighted network games. These two dynamics are similar to the directional learning model (Selten and Stoecker, 1986; Selten and Buchta, 1998) in which players are making attempts to find their targets by adjusting towards the direction they believe the target is located. Such qualitative learning models are known to explain experimental behavior in various settings (Cachon and Camerer, 1996; Cason and Friedman, 1997; Kagel and Levin, 1999; Nagel and Vriend, 1999).

While none of these dynamics can cycle under reciprocal interactions, in directed network games cycles can emerge. Cycles indicate that convergence to the Nash equilibrium is not a universal property of learning processes. We discuss two examples: (1) directed cycle networks allow for best-response cycles as economic activity of the players flows in the opposite direction as the external effects of the network, and (2) parasite-host networks with amplifying links lead to best-response cycles as the parasite's economic activity increases with the host's activity level, while the host's economic activity decreases with that of the parasite. If the weights are assigned by a generic random process such that one-way interactions or parasitic interactions happen with some positive probability, then as the number of players goes to infinity, best-response cycles emerge almost surely.

Nevertheless, classes of networks exist without cycles where convergence to a Nash equilibrium can be shown, given some mild assumptions on the order of updates. In this paper we identify two such classes, networks with transitive relative importance, and games rescalable to exhibit weak influences or weak externalities.

The former class captures networks that can be transformed into symmetric networks with some appropriate rescaling of the action space, an idea raised by Bramoullé et al. (2014). Rescaling can be understood as changing the measurement of one player's production from, e.g., euros to dollars. Rescaling does not affect the equilibrium structure or the convergence properties of the game, but it does change its nominal interaction structure as expressed by the network's weight matrix. Thus, a network with reciprocal interactions can be rescaled into non-reciprocal ones, which thus inherit its convergence properties. A network can be rescaled in such a way if and only if it satisfies the property of transitive relative importance and it does not have one-way or parasitic interactions.

Qualitatively, transitive relative importance means that the relative importance of links must be transitive along players. For instance, if a link between players i and j is more important to i than to j, and if the link between j and a third player k has equal importance to both, then the link between i and k must be more important to i than to k. This property is closely connected to transitive matrices (Farkas et al., 1999) a property applied in pairwise comparison matrices (Bozóki et al., 2010) and the Analytic Hierarchy Processes (Saaty, 1988). We show that these networks and only these can be rescaled into symmetric ones, and these are the only ones that have a quadratic best-response potential function.

The second class of networks with convergent best-response dynamics are those that are rescalable to exhibit weak influences or weak externalities. A player is influenced weakly by her opponents if the total effects of a unit change in all of her opponents' actions on her are smaller than the effect of a unit change in her own action. These networks are characterized by row diagonally dominant weight matrices. In social networks, this property can be interpreted as a form of individualism. On the other hand, weak external effects are characterized by column diagonally dominant weight matrices, meaning a unit change in any player's action has a larger effect on herself than on all the other players combined. In economics, small level of externalities is a characteristic of efficient markets. Convergence to Nash equilibrium in networks with weak influences are studied by Parise and Ozdaglar (2019) while both are studied by Scutari et al. (2014).

Since in our model relations are allowed to be both negative and positive, the two properties are separate from small network effects (Bramoullé et al., 2014; Belhaj et al., 2014). Under small network effects, given an equilibrium production profile, a tremor in a player's production decision is dampened by the network such that the system returns to the original equilibrium. We show that small network effects in absolute value, that is, the spectral radius of the absolute value of the weight matrix being less than one is sufficient and necessary for rescalability into networks with weak influence and those with weak externalities. Moreover every game played on such a network is a best-response potential game, has a unique Nash equilibrium, and all best-response and best-response-approaching dynamics converge to it. While uniqueness of Nash equilibria is known for games played on networks with small network effects, their characterization as rescalable to weak influence or weak externalities networks, the games' potential structure, and the general convergence results are, to our knowledge, new results.

Additionally, this class also includes the set of directed acyclic networks. In this subclass every interaction is one-way and there are no directed cycles. The latter property imposes a hierarchy on the players; players on the highest level of the hierarchy are not affected by any other player, players on intermediate levels are affected by those on higher levels but not by those on lower levels, while players on the lowest level have no effect on any other player. The most direct application of this class of games is pollution management of cities along a river (Ni and Wang, 2007) or a river network (Dong et al., 2012), but the same model is used in studying the conservation of common-pool resources (Richefort and Point, 2010) and games with permission structures (van den Brink et al., 2018).

Overall, our results have a number of implications with respect to convergence in directed network games. A negative finding is that, in networks with directed cycles and parasite-host interactions with amplifying weights, the interpretation of the Nash equilibrium as an outcome of a series of decentralized improvements by the players is questionable as the convergence of simple learning processes is not assured. We complement this with a set of positive results by identifying and characterizing interesting classes of networks where convergence is assured. Our two sets of positive results uncover new insights into the relationship between reciprocity of network interactions, the spectral properties of the network, and the games' potential structure as well as generalize the powerful results achieved for the case of reciprocal interactions.

Our paper is organized as follows: Section 2 presents our two main concepts, directed network games and best-response dynamics. In Section 3 we present the two simple networks leading to the emergence of best-response cycles in large networks. Section 4 contains our characterization and convergence results for networks that can be rescaled into symmetric networks. Section 5 contains the same sets of results for networks rescalable to games with weak influences or weak externalities, and shows that this class includes directed acyclic networks. Section 6 concludes.

2 The model

Let $I = \{1, \ldots, n\}$ be the set of players. For $i \in I$ and upper bounds $\overline{x}_i > 0$ the set $X_i = [0, \overline{x}_i]$ is called the action set of player i, $X = \prod_{i \in I} X_i$ is called the set of action profiles. We let $x_i \in X_i$ denote the action taken by player i while $x_{-i} \in X \setminus X_i$ denotes the truncated action profile of all players except player i, and $x = (x_i)_{i \in I}$ denotes the action profile of all players.

The formal definition of directed network games used in this paper is as follows.

Definition 2.1. The tuple $G = (I, X, (\pi_i)_{i \in I})$ is called a *directed network game* with payoff functions $\pi_i \colon X \to \mathbb{R}$ given by

$$\pi_i(x) = f_i\left(\sum_{j \in I} w_{ij} x_j\right) - c_i x_i, \tag{1}$$

where $f_i : \mathbb{R} \to \mathbb{R}$ is twice differentiable, $f_i' > 0$, $f_i'' < 0$, $w_{ij} \in \mathbb{R}$, $c_i \geq 0$ for every $i \in I$.

The interpretation is the following. Each player produces a specialized good with linear production technology, incurring costs c_i for every unit of the good produced. Players derive benefits from the consumption of their own goods and they are affected by their opponents' production decisions. Player i's enjoyment of player j's good is represented by the weight $w_{ij} \in \mathbb{R}$. We normalize the interaction parameter of each player i with himself, w_{ii}, to 1. The overall benefits of player i are given by the benefit function f_i over the weighted sum of her and her opponents' goods. Crucially, we do not impose reciprocal relations, meaning that $w_{ij} \neq w_{ji}$ may hold, so the weight matrix $(w_{ij})_{i,j \in I} = W$ might not be symmetric.

Since the benefit functions f_i are increasing and concave and the cost parameters c_i are positive, for every x_{-i} there is a unique value of x_i that maximizes $\pi_i(x)$. Let the target values t_i be implicitly defined by $f_i'(t_i) = c_i$, i.e. the value player i would produce if all others produce 0 and let $t = (t_i)_{i \in I}$ denote the vector of targets. We make the simplifying assumption that every player is able to produce her target amount of the good.

Assumption 2.2. We assume that f_i are given such that for every $i \in I$ we have $t_i \in (0, \overline{x}_i)$.

Note that if $w_{ij} > 0$ and $w_{ji} > 0$, then the goods of players i and j are strategic substitutes. If $w_{ij} < 0$ and $w_{ji} < 0$ then their goods are strategic complements. If $w_{ij} > 0$ and $w_{ji} < 0$, then we say that players i and j share a parasitic link. If $w_{ij} = 0$, then player i is not directly affected by player j's production decision.

For player $i \in I$ and $x \in X$, $b_i(x) = \operatorname{argmax}_{x_i \in X_i} \pi_i(x)$ denotes player i's best-response function.

Lemma 2.3. *Let $G = (I, X, (\pi_i)_{i \in I})$ be a weighted network game. Then, for every $i \in I$ and $x \in X$ the best response functions are the following:*

$$b_i(x) = \begin{cases} 0 & \text{if } t_i - \sum_{j \in I \setminus \{i\}} w_{ij} x_j < 0, \\ t_i - \sum_{j \in I \setminus \{i\}} w_{ij} x_j & \text{if } t_i - \sum_{j \in I \setminus \{i\}} w_{ij} x_j \in [0, \overline{x}_i], \\ \overline{x}_i & \text{if } t_i - \sum_{j \in I \setminus \{i\}} w_{ij} x_j > \overline{x}_i. \end{cases} \tag{2}$$

Proof. First, we calculate the unconstrained best response for player i. The first-order condition is

$$\frac{\partial \pi_i(x)}{\partial x} = f_i'\left(\sum_{j \in I} w_{ij} x_j\right) - c_i = 0.$$

Combining with $f_i'(t_i) = c_i$, we get that the unconstrained best response, $\tilde{b}_i(x)$ is

$$\tilde{b}_i(x) = t_i - \sum_{j \in I \setminus \{i\}} w_{ij} x_j. \tag{3}$$

The second-order condition is

$$\frac{\partial^2 \pi_i(x)}{\partial x^2} = f_i''\left(\sum_{j\in I} w_{ij}x_j\right) < 0,$$

hence $\tilde{b}_i(x)$ is indeed maximizing the payoff. This means that for every $x_i > \tilde{b}_i(x)$, a marginal increase of x_i decreases $\pi_i(x)$, while for every $x_i < \tilde{b}_i(x)$, a marginal increase of x_i increases $\pi_i(x)$. Therefore, if $\tilde{b}_i(x) \in [0, \overline{x}_i]$, then $b_i(x) = \tilde{b}_i(x)$. If $\tilde{b}_i(x) < 0$, then $b_i(x) = 0$, as choosing a larger x_i would decrease the payoff. Similarly, if $\tilde{b}_i(x) > \overline{x}_i$, then $b_i(x) = \overline{x}_i$. ■

Lemma 2.4. *Every directed network game has at least one Nash equilibrium.*

Bramoullé et al. (2014)'s analogue result using Brouwer's fixed-point theorem for a positive and symmetric weight matrix is directly applicable in the directed network case. Let the set of Nash equilibria be denoted by X^*.

Lemmas 2.3 and 2.4 have close analogues in existing models with symmetric interaction, but, as we will show shorlty, results concerning the cycling and convergence of simple learning processes do not hold in general in the directed network case.

We now formally introduce the learning processes of our paper, the best-response dynamic and two extensions.

Definition 2.5. The sequence of action profiles $(x^k)_{k\in\mathbb{N}}$ is a *one-sided dynamic* if

- for every $k \in \mathbb{N}$ there exists an $i^k \in I$ such that $x^k_{-i^k} = x^{k+1}_{-i^k}$, and
- for every $i \in I$ the set $K^i = \{k \in \mathbb{N} \colon i^k = i\}$ is infinite.

Definition 2.6 (Best-response dynamics)**.** The one-sided dynamic $(x^k)_{k\in\mathbb{N}}$ is a

- *best-response dynamic* (BRD), if we have $x^{k+1}_{i^k} = b_{i^k}(x^k)$.
- *best-response-approaching dynamic* (BRAD) with approach parameter $0 \leq \beta < 1$, if $|x^{k+1}_{i^k} - b_{i^k}(x^k)| \leq \beta |x^k_{i^k} - b_{i^k}(x^k)|$ and if $x^{k+1}_{i^k} \neq b_{i^k}(x^k)$, then $\operatorname{sgn}(x^{k+1}_{i^k} - b_{i^k}(x^k)) = \operatorname{sgn}(x^k_{i^k} - b_{i^k}(x^k))$.
- *best-response-centered dynamic* (BRCD) with centering parameter $0 \leq \alpha < 1$, if $|x^{k+1}_{i^k} - b_{i^k}(x^k)| \leq \alpha |x^k_{i^k} - b_{i^k}(x^k)|$,

for every $k \in \mathbb{N}$.

In a one-sided dynamic exactly one player changes her action in every time period. In a BRD, every revision takes the updating player to her best response, in a BRAD, players move closer to their best responses without overshooting it, while in a BRCD, players move closer to their best responses and are allowed to overshoot. The approach and centering parameters of a BRAD and a BRCD, respectively, indicate the maximum fraction to which the distances are allowed to decrease. These processes allow payoff-maximizing players to make mistakes, their ability of reaching the best-response is captured by the two parameters with lower values indicating a higher level of accuracy. It is clear that every BRD is a BRAD and every BRAD is a BRCD.

In all cases we restrict our attention to dynamics where every player revises infinitely many times. This brings the property that every convergent dynamic will converge to a Nash equilibrium, and no player can get stuck playing a suboptimal action indefinitely by not having the opportunity to revise.

Bramoullé et al. (2014) and Bervoets and Faure (2019) consider the BRD in continuous time. Parise and Ozdaglar (2019), in addition to the continuous-time dynamic, also considers discrete-time BRD, both simultaneous (all players update in every period) and sequential (one-sided, players update in a fixed order repeating every n periods). Our process is more general than the latter as revision opportunities may arrive in any order. Bayer et al. (2019b) considers both the BRD and the BRCD as above.

Finally in this section we define one of the main concepts used in this paper, best-response potential games.

Definition 2.7 (Voorneveld (2000)). A game $G = (I, X, (\pi_i)_{i\in I})$ is a *best-response potential game*, if there exists a *best-response potential function* $\phi : X \to \mathbb{R}$ such that for every $i \in I$, and every $x_{-i} \in X_{-i}$ it holds that

$$\operatorname*{argmax}_{x_i \in X_i} \pi_i(x) = \operatorname*{argmax}_{x_i \in X_i} \phi(x).$$

Definition 2.7 states that G is a best-response potential game if the best-response behavior of all players can be characterized by a single real-valued function ϕ, called the best-response potential.

3 Networks with best-response cycles

In this section we discuss the cycling of best-response dynamics. The non-existence of best-response cycles is a necessary but not sufficient condition of the convergence of best-response dynamics (Kukushkin, 2015), which is true in networks with reciprocal interactions. In this section we show that directed cycle networks and parasitic relations with amplifying links lead to cycling and thus hinder any general convergence results. We then show that, for large networks with a probabilistic assignment of weights, best-response cycles exist almost surely.

We begin with a formal definition of cycles.

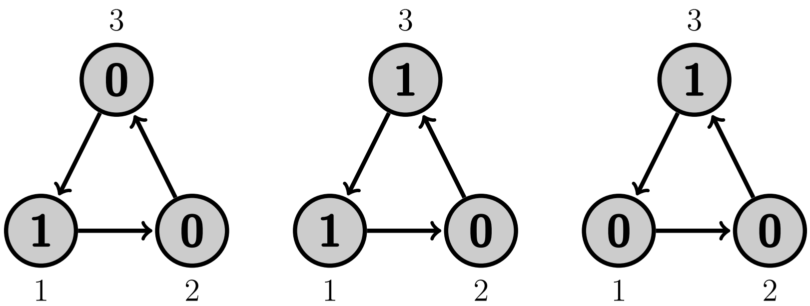
Definition 3.1 (Cycles)**.** A sequence $(x^k)_{k\in\mathbb{N}}$ has a *cycle* if there exist three time periods, $k < k' < k''$ such that $x^k = x^{k''}$, but $x^k \neq x^{k'}$.

In words, a process has a cycle if it non-trivially revisits an action profile in two different time periods.

Example 3.2 (Directed cycle network)**.** Consider the directed cycle network, with $I = \{1, 2, 3\}$, $X_i = [0, 1]$ for $i \in I$, $t = (1, 1, 1)^\top$ and the weight matrix

$$W = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

representing a directed cycle with three players. This game has a single Nash equilibrium, $x^* = (0.5, 0.5, 0.5)^\top$. Consider the BRD with the initial action profile $x = (1, 0, 0)^\top$. Let player 3 receive the first revision opportunity, followed by player 1.



Period 1. Period 2. Period 3.

Figure 1: Economic activity shifts in the opposite direction of externalities under the BRD.

Figure 1 shows how player 1's initial production moves along its incoming link to player 3. It is easy to see that by allowing player 2 to revise next followed by player 3, activity shifts further on to player 2. Moving along one step further completes the best-response cycle to $x = (1, 0, 0)^\top$.

The directed cycle of this example thus permits a best-response cycle in which the players' activity moves along the cycle in the opposite direction as the players' externalities. It is easy to see that this property remains true for directed cycles of n players. If n is even, it may be possible to reach an equilibrium in which every second player produces 1 and the others produces 0 for some order of revisions, but no 0-1 equilibrium exists if n is odd. Therefore, in the latter case, if every player begins at a production level of either 0 or 1, then the only two possible production levels in any best-response path are also 0 and 1, and hence, convergence to the Nash equilibrium is impossible.

Next, we consider a parasitic link between two players.

Example 3.3 (Parasitism)**.** Let $I = \{1, 2\}$, $t_1, t_2 = 1$ and let

$$W = \begin{pmatrix} 1 & -2 \\ 0.5 & 1 \end{pmatrix}.$$

Under the BRD, starting from the action profile $x = (1, 0)^\top$, if both players revise in turns, the game has a best-response cycle of length four.

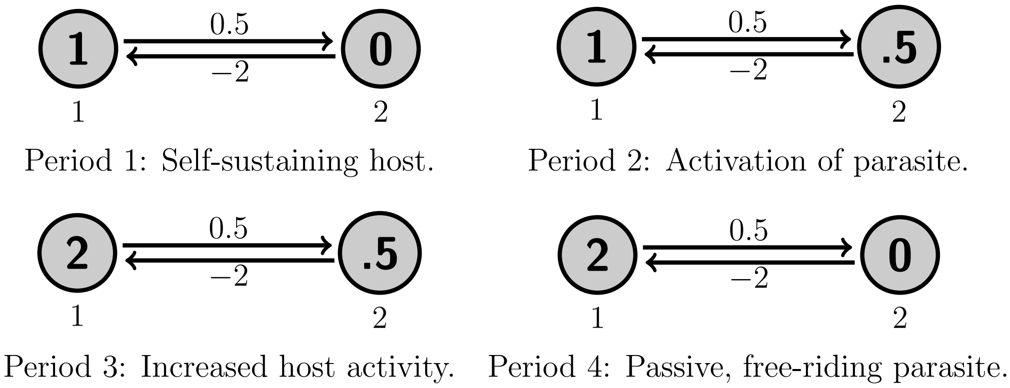


Figure 2: The best-response cycle of Example 3.3.

In this example, player 1 is called the host and player 2 is called the parasite. A self-sustaining host is engaged by a parasite. The host responds by increasing her activity to offset the parasite's negative effects. The parasite's benefits from the host are large enough that it ceases production entirely and free-rides on the host. The host is then able to return to the self-sustaining stage, completing the cycle. This process is shown in Figure 2.

The same pattern of parasite-host interaction can be replicated by different calibration of parameters. A parasitic network requires $w_{12} < 0$ and $w_{21} > 0$. If the network weights and targets also satisfy

(i) $\overline{x}_1 \geq t_1 - w_{12}(t_2 - w_{21}t_1)$ (host has a large enough capacity to feed the parasite),

(ii) $\overline{x}_2 \geq t_2 - w_{21}t_1$ (parasite has a large enough capacity to cause a nuisance to a self-sustaining host),

(iii) $t_2 - w_{21}t_1 > 0$ (self-sustaining host does not satisfy the parasite),

(iv) $|w_{12}w_{21}| \geq 1$ (the link is amplifying),

then the best-response cycle qualitatively equivalent to that of Figure 2 is achieved by sequential updates by the players.

Examples 3.2 and 3.3 together indicate that directed cycles and parasitic relations lead to the cycling of BRDs, and hence that of BRADs and BRCDs. The remainder of this section shows that if the (ex-ante) possibility of one-way or parasitic interactions exists in an interaction network, even if the possibilities are very small, then large networks will have a best-response cycle almost surely. The intuition for this is quite simple: With randomized player interactions directed cycles or parasitic links producing the cycles seen in Examples 3.2 and 3.3 may appear with positive probability. As the number of players goes to infinity, a network allowing for best-response cycles will appear almost surely.

We now turn to a formal description of the above idea. For $i, j \in I$ with $i \neq j$ let the weight w_{ij} be the realization of a random variable denoted by $\tilde{w}_{ij}$.

Assumption 3.4. For every $i, i', j, j' \in I$ such that $\{i, j\} \neq \{i', j'\}$ the variables $\tilde{w}_{ij}$ and $\tilde{w}_{i'j'}$ are independent.

Assumption 3.4 states that any two distributions producing weights not belonging to the same pair of players are independent. This assumption is made mostly for convenience, our results may be obtained with less stringent assumptions. Crucially, we allow the two weights describing the interaction between a pair of players to be dependent.

Let $w^-, \underline{w}, \overline{w} > 0$ be given such that $\underline{w} < \overline{w}$. We introduce the following notation:

1. $P_0 = \min_{i,j \in I}(P(\tilde{w}_{ij} = \tilde{w}_{ji} = 0))$,

2. $P_1(\underline{w}) = \min_{i,j \in I}(\min\{P(\tilde{w}_{ij} \geq \underline{w}, \tilde{w}_{ji} = 0), P(\tilde{w}_{ji} \geq \underline{w}, \tilde{w}_{ij} = 0)\})$.

3. $P_2(w^-, \underline{w}, \overline{w}) = \min_{i,j \in I}(\min\{P(\tilde{w}_{ij} \in [\underline{w}, \overline{w}], \tilde{w}_{ji} \leq -w^-), P(\tilde{w}_{ji} \in [\underline{w}, \overline{w}], \tilde{w}_{ij} \leq -w^-)\})$,

In words, the smallest probability that two players are indifferent to one another is denoted by P_0, the smallest probability of there being a one-directional link of at least strength $\underline{w}$ between an ordered pair is denoted by $P_1(\underline{w})$, and the smallest probability that an ordered pair has a parasitic link such that the parasite's effect on the host is at least w^- and the host's effect on the parasite falls into the interval $[\underline{w}, \overline{w}]$ is denoted by $P_2(w^-, \underline{w}, \overline{w})$.

Definition 3.5. We say that the random process generating a network satisfies/allows for

- *no forced interaction* (NFI) if $P^0 > 0$,

- *directed interactions* (DI) if there exists $\underline{w} > 0$ such that $P_1(\underline{w}) > 0$.

- *biparasitism* (BP) if there exist $w^-, \underline{w}, \overline{w} > 0$ such that $P_2(w^-, \underline{w}, \overline{w}) > 0$, The properties listed in Definition 3.5 are interpreted as follows: *NFI* means that any two players have a positive probability of mutual indifference. *DI* means that every player may unilaterally influence any other with positive probability, a possibility of one-way indifference. *BP* means that any player may be a parasite of any other with positive probability, an ex-ante possibility of parasitism between any ordered pair. Notice that the three properties impose no restriction on the ex-post realizations of weights. Since all three numbers can be arbitrarily small, the ex-ante restriction is also mild.

Theorem 3.6. *As n goes to infinity, if the random process generating the network satisfies* NFI *and* DI*, then the resulting directed network game allows for a best-response cycle almost surely.*

The proof is shown as an appendix.

Theorem 3.7. *For $i \in I$ let $\overline{x}_i = \infty$.*

As n goes to infinity, if the random process generating the network satisfies NFI *and* BP *such that $\overline{w} < \min_{i\in I} t_i / \max_{i\in I} t_i$, then the resulting directed network game allows for a best-response cycle almost surely.*

The proof is shown as an appendix.

Theorems 3.6 and 3.7 establish that, given *NFI*, either of *BP* and *DI* are enough to ensure that large networks will allow for best-response cycles with some additional assumptions on the parameters in the latter case. These properties are mild, in terms of the restrictions they impose on the probability distributions on the weights, as well as intuitive, in terms of what they mean for the links between the players. In large networks, even small likelihood of one-way interactions or parasitic interactions lead to the cycles similar to Examples 3.2 and 3.3 almost surely. Thus we identify directed cycles and parasitism as important stumbling blocks of convergence. Both theorems generalize to non-independent distributions of weights between different pairs of players if the probabilities defined in Definition 3.5 are positive, conditional on the possible realizations of other weights.

4 Transitive relative importance

As shown in the previous section, allowing for non-reciprocal interactions in network games changes their convergence properties under BRDs. However, there are classes of directed network games where convergence to the game's Nash equilibrium can still be shown.

It is well known in the literature that convergence of BRDs in networks of reciprocal interactions can be shown by exploiting the game's potential structure. In section VI, Bramoullé et al. (2014) raises the idea that, by an appropriate rescaling of the action space, this method may be extended for some class of directed networks as well. In this section we identify and characterize this class of networks as those with transitive weight matrices. Additionally, we show that, of all directed network games, games played on these and only these networks have quadratic best-response potentials.

We first formally restate the convergence result in the symmetric case for future use. We begin by defining regular updates by players.

Definition 4.1. We say that players *update regularly* in a one-sided dynamic $(x^k)_{k\in\mathbb{N}}$, if there exists a $K > 0$ such that for every $i \in I$ and $k \in \mathbb{N}$ there exists a k' such that $k < k' \leq k + K$ and $i^{k'} = i$.

Players updating regularly ensures that no player's frequency of receiving revision opportunities approaches zero.

Lemma 4.2. *Let W be given such that for every $i, j \in I$ we have $w_{ij} = w_{ji}$ and let t be such that $|X^*| < \infty$. Then, every BRD and BRCD $(x^k)_{k\in\mathbb{N}}$ in which players update regularly converges to a Nash equilibrium.*

For the full proof or Lemma 4.2 see Theorem 5.3 in Bayer et al. (2019b). By Lemma 3.2 in Bayer et al. (2019b), for every network, the set of target vectors which result in infinitely many Nash equilibria is small both in the measure theoretic sense and the topologial sense, so the finiteness condition is not restrictive. The main intuition for the BRD part is that $w_{ij} = w_{ji}$ implies that the function

$$x^\top t - \frac{1}{2}x^\top Wx$$

serves as a best-response potential. Every player's update will weakly increase the value of the best-response potential, which is bounded as the action space is bounded. Regular updating ensures that, in time, every player will be close to her current best-response, while finite number of Nash equilibria ensures that convergence of ϕ translates to convergence to an isolated peak of the potential value landscape, corresponding to a Nash equilibrium. Additionally, Bayer et al. (2019b) show that the finiteness of the equilibrium set is a generic property of undirected network games.

Take a vector $a \in R^n$, called a scaling vector, such that $a > 0$, i.e., for every $i \in I$ we have $a_i > 0$ and let $y_i = a_i x_i$ and $\overline{y}_i = a_i \overline{x}_i$. Furthermore, let $Y_i = [0, \overline{y}_i]$ and $Y = \prod_{i\in I} Y_i$. It is clear that any BRD, BRAD, or BRCD, respectively, in the game played on X is a BRD, BRAD, or

BRCD, respectively, in the game played on Y with the same approach or centering parameters in the cases of BRAD and BRCD. Similarly, the convergence properties of the processes in the game played on Y are identical to those in the game played on X.

By Lemma 2.3 the unconstrained best-response function of player i in the rescaled game is given by

$$t_i a_i - \sum_{j \in I \setminus \{i\}} w_{ij} \frac{a_i}{a_j} y_j.$$

Let $w_{ij} a_i / a_j = v_{ij}$ and let $V = (v_{ij})_{i,j \in I}$ denote the matrix of rescaled weights. Our goal in this section is to characterize the class of networks that are rescalable into a symmetric network, i.e. the set of networks W for which there exists a vector $a > 0$ such that for every $i, j \in I$ we have

$$\frac{a_i}{a_j} w_{ij} = v_{ij} = v_{ji} = \frac{a_j}{a_i} w_{ji}. \tag{4}$$

We begin with the definition of transitive relative importance.

Definition 4.3. We say that the weight matrix W shows *transitive relative importance* if for every $3 \leq m \leq n$ and for all pairwise distinct $i_1, i_2, \ldots, i_m \in I$ we have

$$w_{i_1 i_2} w_{i_2 i_3} \ldots w_{i_{m-1} i_m} w_{i_m i_1} = w_{i_1 i_m} w_{i_m i_{m-1}} \ldots w_{i_3 i_2} w_{i_2 i_1}.$$

For simplicity we write that W is transitive if it satisfies Definition 4.3. Notice that if W is symmetric, then it is also transitive. In fact, a symmetric W satisfies the requirements of Definition 4.3 for $2 \leq m \leq n$. Also notice that if for every $i, j \in I$ it holds that $w_{ij} \neq 0$, then $w_{ij} w_{jk} w_{ki} = w_{ik} w_{kj} w_{ji}$ for every pairwise distinct $i, j, k \in I$ implies transitivity.

The interpretation is as follows: for $w_{ij}, w_{ji} \neq 0$ define the value $r_{ij} = w_{ij} / w_{ji}$ as player i's relative importance on the link between i and j. It is clear that, if well-defined, the matrix $R = (r_{ij})_{i,j \in I}$ is symmetrically reciprocal. In this case the transitivity property reduces to having

$$r_{ij} r_{jk} = r_{ik}, \tag{5}$$

for every $i, j, k \in I$. Thus, qualitatively, if the link between i and j is more important to i than to j and if the link between j and to k is more important to j than k, then the link between i and k must be more important to i than to k. Note that condition (5) is identical to the condition imposed on consistent pairwise comparison matrices in Analytic Hierarchical Processes (Saaty, 1988). Here, the literature seems to use the terms consistent (Bozóki et al., 2010) and transitive

(Farkas et al., 1999) interchangeably when describing the symmetrically reciprocal comparison matrix R.

The final definition we require is that of sign-symmetry.

Definition 4.4. We say that the weight matrix W is sign-symmetric if for every $\{i, j\} \subseteq I$ we have $\text{sgn}(w_{ij}) = \text{sgn}(w_{ji})$.

Sign-symmetry of networks rules out one-way interactions and parasitic interactions between players.

We are ready to present the main result of this section.

Theorem 4.5. *The following statements are equivalent:*

1. *the network W is rescalable into a symmetric matrix,*
2. *the network W is transitive and sign-symmetric,*
3. *there exists a game G on W that has a quadratic best-response potential function,*
4. *every game G on W has a quadratic best-response potential function.*

The proof is shown as an appendix.

Theorem 4.5 shows that transitivity of a network combined with sign-symmetry is equivalent to it being rescalable into a symmetric network. Additionally, no other network has a quadratic potential function, which shows that this property cannot be exploited further. Our result thus gives a full characterization for which types of directed networks satisfy the requirements put forward in Bramoullé et al. (2014) section VI.

Theorem 4.5 and Lemma 4.2 give rise to the following corollary.

Corollary 4.6. *Let W be a transitive and sign-symmetric network and let t be given such that $|X^*| < \infty$. Then, every BRD and BRCD in which players update regularly converges to a Nash equilibrium.*

For a fixed symmetric network, since the number of Nash equilibria is finite for almost every target vector (Bayer et al., 2019b), Corollary 4.6 also applies to every network and almost every target vector, thus convergence is generically established for this class of networks.

k	x_1^k	x_2^k	x_3^k	i^k	$\sum_{j \in I} w_{i^k j} x_j$	$b_{i^k}(x_{i^k})$
0	1	0	0	3	0	1
1	1	0	1	1	δ	$1-\delta$
2	$1-\delta$	0	1	2	$\delta-\delta^2$	$1-\delta+\delta^2$
3	$1-\delta$	$1-\delta+\delta^2$	1	3	$\delta-\delta^2+\delta^3$	$1-\delta+\delta^2-\delta^3$

Table 1: The best-response dynamic of Example 5.1.

5 Weak influences and weak externalities

In this section we characterize another class of networks with convergent dynamics. A key concept in describing this class is the players' influence and the externalities they produce. A player i's decisions are influenced by her opponents through her incoming weights, measured by their total magnitude: $\sum_{j \in I \setminus \{i\}} |w_{ij}|$. Similarly, a player i's external effects on her opponents is measured by the total magnitude of her outgoing weights: $\sum_{j \in I \setminus \{i\}} |w_{ji}|$.

In this section we show that if total influences or externalities are weak enough in a network, then games played on this network have a unique Nash equilibrium and all BRDs and BRADs converge to it. As a motivating example, consider a parametric version of Example 3.2.

Example 5.1 (Directed cycle with weights)**.** Consider the three player directed cycle network given by the weight matrix

$$W = \begin{pmatrix} 1 & 0 & \delta \\ \delta & 1 & 0 \\ 0 & \delta & 1 \end{pmatrix},$$

and with $\delta \in [0,1]$. By Lemma 2.3, the best-response functions are $b_1(x) = 1 - \delta x_3$, $b_2(x) = 1 - \delta x_1$, $b_3(x) = 1 - \delta x_2$. The only Nash equilibrium is $x^* = (1/(1+\delta), 1/(1+\delta), 1/(1+\delta))^\top$.

If the strength of the influences is as strong as the own effects, i.e. $\delta = 1$, then best-response cycles may exist as shown in Example 3.2. In a game with no interaction, $\delta = 0$, best-response cycles cannot exist, since the matrix is symmetric. We now consider how the cycling properties change by changing δ.

In Table 1 we show the sequence of action profiles in the BRD where players receive revision opportunities in the same, repeating order $(3,1,2)$, starting, again, in the action profile $(1,0,0)^\top$. In this order of revisions, the player holding the revision opportunity in period k will revise to $\sum_{\ell=0}^{k} (-\delta)^\ell = (1-(-\delta)^{k+1})/(1+\delta)$ for parameter values of $\delta < 1$ (see Table 1). Playing on in this order will produce no cycles, and lead to convergence to the Nash equilibrium.

Example 5.1 suggests that even a non-negligible level of influences/externalities can lead to the disappearance of best-response cycles. As we will show in this section, this turns out to be a general property: if every player's total influences or total externalities are smaller than her own weight, the game has a single Nash equilibrium and every BRD and BRAD converges to it.

We introduce these games formally.

Definition 5.2. A network W has

- *weak influences* if for every $i \in I$ it holds that $\sum_{j \in I \setminus \{i\}} |w_{ij}| < 1$,
- *weak externalities* if for every $i \in I$ it holds that $\sum_{j \in I \setminus \{i\}} |w_{ji}| < 1$.

Networks with weak influences are characterized by *row diagonally dominant* weight matrices, while those with weak externalities have *column diagonally dominant* weight matrices. Games with weak influences satisfy Assumption 2b of Parise and Ozdaglar (2019), while both classes are covered by Scutari et al. (2014)'s Proposition 7.

We show that both classes of games have a unique Nash equilibrium and every BRD and BRAD converges to it. Once again we can make use of rescaling: any network which is rescalable into one of the two classes inherits the uniqueness of the Nash equilibrium as well as the convergence properties. As before, for $a \in \mathbb{R}^n$, $a > 0$, define $V = (v_{ij})_{i,j \in I}$ as $v_{ij} = w_{ij} a_i / a_j$. It turns out that the two classes are rescalable into each other. Furthermore, a network is rescalable to either class if and only if the spectral radius of $|W| - I_n$ is less than one, where I_n is the $n \times n$ identity matrix.

Recall that the *spectral radius* $\rho(M)$ of a square matrix $M \in \mathbb{C}^{n \times n}$ is the largest absolute value of its eigenvalues, i.e.,

$$\rho(M) = \max\{|\lambda| : \lambda \in \mathbb{C} \text{ is an eigenvalue of } M\}.$$

The next proposition states that a network W can be rescaled into one with weak influences and one with weak externalities if and only if $\rho(|W| - I_n) < 1$.

Proposition 5.3. *The following statements are equivalent for W.*

1. *There exists a scaling vector $a \in \mathbb{R}^n$, $a > 0$ such that the rescaled matrix V has weak influences.*
2. *There exists a scaling vector $a \in \mathbb{R}^n$, $a > 0$ such that the rescaled network V has weak externalities.*
3. $\lim_{k \to \infty} (|W| - I_n)^k = 0$.
4. $\rho(|W| - I_n) < 1$.

We follow by showing that games played on such networks have a unique Nash equilibrium and are best-response potential games.

Proposition 5.4. *If $\rho(|W| - I_n) < 1$, then every game played on the network W has a unique Nash equilibrium.*

Proposition 5.5. *Consider a game played on a network W, and assume that $a \in \mathbb{R}^n$, $a > 0$ is a scaling vector such that the rescaled network V has weak externalities. Then, the function*

$$\phi'(x) = -\sum_{i \in I} a_i |x_i - b_i(x)|$$

is a best-response potential function of the game.

The main result in this section is the convergence of BRD and BRAD in this gameclass.

Theorem 5.6. *If $\rho(|W| - I_n) < 1$, then in every game played on network W, every BRD and BRAD converges to the unique Nash equilibrium.*

All four proofs are shown as an appendix.

As demonstrated by Example 3.2, a spectral radius equal to 1 leads to the emergence of best-response cycles, hence these results are tight. Additionally, notice that the BRCD may lead to cycles in this gameclass. For instance, the BRD shown in Example 3.2 is a BRCD in Example 5.1 for $\delta = 0.9$.

It is easy to show that a necessary condition of a network being rescalable to one with weak externalities is that no pair of players have an amplifying link, i.e. one where the product of weights is larger than the size of the own effects in absolute value.

Lemma 5.7. *Let W be given such that there exists an $a \in \mathbb{R}^n$, $a > 0$ for which the rescaled network V is with weak externalities. Then, for every $i, j \in I$ it holds that $|w_{ij}w_{ji}| < 1$.*

Proof. Suppose that $|w_{ij}w_{ji}| > 1$. Since V has weak externalities we must have $v_{ij}, v_{ji} < 1$. However,

$$|v_{ij}v_{ji}| = |\frac{a_i}{a_j} w_{ij} \frac{a_j}{a_i} w_{ji}| > 1,$$

a contradiction. ■

By Definition 5.2 it is clear that unlike networks that are rescalable to be symmetric, those that are rescalable to exhibit weak influences or externalities may allow one-way interactions as well as parasitic ones. However, in the latter case, the link cannot be amplifying (Lemma 5.7), which rules out the best-response cycles seen in Example 3.3. Additionally, as seen in Example 3.2, if one-way interactions form cycles we once again get best-response cycles.

In the absence of cycles in a network, it can always be rescaled into one with weak externalities. We now formalize this idea starting with the definition of directed acyclic networks.

Definition 5.8. A network W is called a *directed acyclic network (DAN)* if for every $i < j$, $i, j \in I$ we have $w_{ij} = 0$.

In other words, if W is lower triangular, then the game is played on a *DAN*. One can think of an equivalent characterization with upper triangular matrices, or any permutation of players which leaves W as a lower triangular matrix. In Definition 5.8 players with lower indices are higher up in the hierarchy, i.e. player 1 is unaffected by any other player's action, player 2 is only affected by player 1, etc.

An economic application of this game is pollution management of a number of cities with industrial zones located along a river. Each city decides on the amount of money spent on cleaning the industrial waste in the river and their decisions affect only those cities that are located downstream. The cities' target values describe the point at which the marginal benefits of an extra dollar's worth of cleaner water are the same as the costs for that city. Models of this problem include Ni and Wang (2007) and Dong et al. (2012).

Games of the above nature introduce a hierarchy of players. Such hierarchies are present in most production chains where goods – and therefore externalities – flow downstream, or in some organizational networks such as the military where orders are traveling down the chain of command. Directed acyclic cycles have applications in biology as well among many other fields; most trophic networks also have hierarchical features with apex predators on the highest level and prey animals on lower levels.

We now show that every game played on a DAN is rescalable into one with weak externalities.

Proposition 5.9. *For every DAN, W there exists $a \in \mathbb{R}^n$, $a > 0$ such that the rescaled network, V has weak externalities.*

Proof. Let W be a DAN, i.e., we have $w_{ij} = 0$ for every $i < j$. First, let $a_n = 1$. We will define the rest of the a_i's recursively. Assume that we have already defined a_n, $a_{n-1}, \dots, a_{n-j+1}$. Let us choose $a_j > 0$ so that we have

$$\sum_{k=j+1}^{n} a_k |w_{kj}| < a_j. \tag{6}$$

Now we can verify that the rescaled matrix V has weak externalities. Let $j \in I$. We have

$$\begin{aligned}
\sum_{i \in I \setminus \{j\}} |v_{ij}| &= \sum_{i \in I \setminus \{j\}} |w_{ij}| \frac{a_i}{a_j} \\
&= \sum_{i=1}^{j-1} 0 \cdot \frac{a_i}{a_j} + \frac{1}{a_j} \sum_{i=j+1}^{n} |w_{ij}| a_i \\
&< 0 + \frac{1}{a_j} \cdot a_j = 1 \qquad \text{by (6).}
\end{aligned}$$

This concludes the proof. ■

The following corollary regarding the convergence of BRD and BRAD is implied by Theorem 5.6.

Corollary 5.10. *For every game played on a DAN, every BRD and BRAD converges to the game's unique Nash equilibrium.*

6 Conclusion

In this paper we analyze directed network games, a generalization of the private provision of public goods games model to include possibly non-reciprocal relationships. These cover one-way interactions, unequal interactions, and parasitism. While weighted networks and simple graphs are very useful frameworks, more nuanced models of social and economic networks should include non-reciprocal interactions.

While best-response dynamics on games played on symmetric networks are known to converge to a Nash equilibrium due to the games' potential structure, this is not true in general for networks with asymmetric weight matrices. In this paper we show that both one-way interactions and parasitic interactions can create best-response cycles. In a setting where interaction weights are assigned randomly, mild assumptions on the random process lead to best-response cycles almost surely in large networks. This questions the interpretation of the Nash equilibrium as the result of individual improvements by the players. Together with other known problems of the Nash equilibrium both conceptual and behavioral, equilibrium analysis of such games may be of questionable value in settings with possibly non-reciprocal interactions.

There are classes of asymmetric networks, however, where the predictive power of the Nash equilibrium is retained. In this paper we highlight two such classes; those that can be rescaled into symmetric networks and those that are rescalable to networks with weak influences or weak externalities. We characterize the former type by transitive relative importance of players and sign-symmetry of the weight matrix. Additionally, this class captures all networks with quadratic best-response potential functions, indicating that other network types with convergence require different approaches to identify.

The latter class captures individualistic social networks as well as situations where the economic externalities have been internalized. We show that these types are equivalent with respect to rescaling and any network with a spectral radius less than one is rescalable to either. Such games are best-response potential games, have a unique Nash equilibrium, and all BRDs and BRADs converge to it. A necessary condition for a network to be rescalable to one with weak externalities is the absence of amplifying links.

Directed acyclic networks can always be rescaled into networks with weak externalities. This subclass imposes a hierarchy on the players; players are only influenced by opponents on higher levels and only influence those on lower levels. Directed acyclic networks have an established application in economics in the pollution management of cities along a river or a river network, but, since the hierarchical structure is widely studied, there are other potential fields of application.

Our results unlock a number of insights into network games. The most apparent general result is a negative one: the convergence properties of games played on symmetric networks do not generalize well for the asymmetric case. For directed cycles and parasitic interactions best-response cycles may appear, thus identifying convergent classes of networks that include any of these types of interactions are likely to require different methodologies than the best-response potentials, rescaling, and spectral properties used in this paper.

Our positive contribution consists of the full exploration of the idea of rescalability into symmetric matrices as well as the identification of weak influences/externalities as networks with convergent dynamics and the full characterization of the latter class. On a technical level we identify all network games with a quadratic potential structure and unlock a new class with a different potential structure. We thus broaden the set of sufficient conditions that guarantee convergence in network games. Finding a set of sufficient and necessary conditions, or, failing that, broader sets of sufficient conditions, is an important direction left for future research.

A Appendix: Proofs

Proofs for Section 3

Let $\underline{t} = \min_{i \in I} t_i$ and let $\bar{t} = \max_{i \in I} t_i$. For a fixed $w^-, \underline{w}, \overline{w}$, we shorten the notations $P_1(\underline{w})$ and $P_2(w^-, \underline{w}, \overline{w})$ to P_1 and P_2, respectively.

Theorem 3.6

Proof of Theorem 3.6. Let m be such that $mt\underline{w} \geq \bar{t}$. For some $\ell \in \mathbb{N}$ fix n such that $n = 3m\ell$.

We construct three sets of m players, I_1, I_2, and I_3, such that

- for every pair of players i, j, $i \neq j$, belonging to the same group, $w_{ij} = w_{ji} = 0$, and
- for every trio of players $i^1 \in I^1$, $i^2 \in I^2$, $i^3 \in I^3$, it holds that $w_{i^1 i^2} = w_{i^2 i^3} = w_{i^3 i^1} = 0$, $w_{i^1 i^3}, w_{i^2 i^1}, w_{i^3 i^2} > \underline{w}$.

The probability of such realization of weights occurring is at least $P_0^{3m(m-1)/2} P_1^{3m^2}$ which is positive by properties *NFI* and *DI*.

We now show that, given such a group of players, no matter the realization of any other weights in the network, the game has a best-response cycle.

Let $x^1, x^2, x^3 \in X$ be given as follows: For $j \in \{1, 2, 3\}$ $x_i^j = 0$ if $i \notin I^j$ and $x_i^j = t_i$ if $i \in I^j$. Let $x^{-1}, x^{-2}, x^{-3} \in X$ be given as follows: for every $i \in I$ $x_i^{-1} = \max\{x_i^2, x_i^3\}$, $x_i^{-2} = \max\{x_i^1, x_i^3\}$, $x_i^{-3} = \max\{x_i^1, x_i^2\}$.

In profile x^1, only the members of group I^1 contribute a positive amount, every other player contributes zero. Since $ms\underline{w} \geq \bar{t}$, all members of group I^2 are at best response. Since the members of group I^3 receive 0, the profile x^{-2} follows x^1 if all members of I^3 move to their best responses in any order. Since $ms\underline{w} \geq mt\underline{w} \geq \bar{t}$, the best response of the members of group I^1 to x^{-2} is to move to 0. Hence the profile x^3 follows x^{-2} by the members of I^1 moving to their best responses in any order. Very similarly, x^{-1} follows x^3, x^2 follows x^{-1}, x^{-3} follows x^2, and x^1 follows x^{-3}, completing the cycle.

The probability of three such groups forming in a population of $3m\ell$ players is at least

$$1 - (1 - P_0^{3m(m-1)/2} P_1^{3m^2})^{\ell}.$$

Let E_n denote the event that a best-response cycle exists in a game of n players. For every $n \geq 0$ it is clear that $P(E_n) \leq P(E_{n+1}) \leq 1$, so the sequence $P(E_n)$ is convergent. Therefore, we have

$$\lim_{n\to\infty} P(E_n) = \lim_{\ell\to\infty} P(E_{3m\ell}) \geq 1 - \lim_{\ell\to\infty} (1 - P_0^{3m(m-1)/2} P_1^{3m^2})^{\ell} = 1.$$

■

Theorem 3.7

Proof of Theorem 3.7. We introduce the notation $z = \underline{t} - \overline{w}\bar{t}$. Notice that $z > 0$ by the assumption of the Theorem. Fix $m \in \mathbb{N}$ such that $\bar{t} - \underline{wt} < m\underline{w}z$.

We construct a set of m players, I_1, and a player $j \in I \setminus I_1$ such that

- for every pair of players $i, i' \in I_1$, $i \neq i'$, we have $w_{ii'} = w_{i'i} = 0$, and
- for every $i \in I_1$ it holds that $w_{ij} \in [\underline{w}, \overline{w}]$ and $w_{ji} \leq -w^-$.

The probability of such a realization of weights happening between $m+1$ players is at least $P_0^{m(m-1)/2} P_2^m$, which is positive due to properties *NFI* and *BP*.

We now show that, given such a group of players, no matter the realization of any other weights in the network, the game has a best-response cycle. For $i \in I_1$ let $z_i = t_i - w_{i1} t_j$. Notice that $z_i \geq z > 0$. Let x^1, x^2, x^3, x^4 be given as follows: For every $i \notin I_1 \cup \{j\}$ $x_i^1 = x_i^2 = x_i^3 = x_i^4 = 0$, let $x_j^1 = x_j^2 = t_j$ and $x_j^3 = x_j^4 = t_j - w_{ji} z_i$, while for $i \in I_1$ let $x_i^1 = x_i^4 = 0$ and $x_i^2 = x_i^3 = z_i$.

In profile x^1, player j contributes her target, the members of I^1 contribute zero. Since $b_i(x^1) = z_i$, a non-negative value by $z_j \geq z > 0$, the profile x^2 follows from x^1 by the members of I^1 moving to best response in any order. Since $\overline{x}_j = \infty$, it also holds that $b_j(x^2) = t_j - \sum_{i \in I^1} w_{ji} z_i$, a non-negative value by $w_{ji} \leq -w^- < 0$, hence the profile x^3 follows from x^2 by j moving to her best response. For $i \in I_1$ we have

$$\hat{b}_i(x^3) = z_i + w_{ij} \sum_{i' \in I^1} w_{ji'} z_{i'} \leq \overline{t} - \underline{wt} - m\underline{w}z < 0$$

by the choice of m. Hence, for $i \in I^1$ we have $b_i(x^3) = 0$ and x^4 follows x^3 by the members of I^1 moving to best response in any order. Finally, x^1 follows x^4 by player j moving to her best response, completing the cycle.

The probability of such groups forming in a population of $(m+1)\ell$ players is at least

$$1 - (1 - P_0^{m(m-1)/2} P_2^m)^\ell. \tag{7}$$

Let E_n denote the event that a best-response cycle exists in a game of n players. For every $n \geq 0$ it is clear that $P(E_n) \leq P(E_{n+1}) \leq 1$, hence the sequence $P(E_n)$ is convergent. Then we have

$$\lim_{n \to \infty} P(E_n) = \lim_{\ell \to \infty} P(E_{(m+1)\ell}) \geq 1 - \lim_{\ell \to \infty} (1 - P_0^{m(m-1)/2} P_2^m)^\ell = 1.$$

■

Proofs for Section 4

Theorem 4.5

It is clear that statement 4 implies statement 3. We show the remaining three implications.

Proof of 1 ⇒ 4. Suppose that the matrix W can be rescaled into a symmetric matrix $(v_{ij})_{i,j\in I}$ with the vector $a \in \mathbb{R}^n$, $a > 0$, i.e., we have $v_{ij} = w_{ij}a_i/a_j$. Let G be any game on W. We now show that the following quadratic function is a best-response potential of the game.

$$\phi^Q(x) = \sum_{i\in I} a_i^2 x_i t_i - \frac{1}{2}\sum_{i\in I}\sum_{j\in I} x_i x_j a_i a_j v_{ij}.$$

For every $i \in I$, the partial derivative is as follows:

$$\begin{aligned}\frac{\partial\phi^Q}{\partial x_i}(x) &= a_i^2 t_i - \sum_{j\in I} x_j a_j a_i v_{ij}\\ &= a_i^2 t_i - \sum_{j\in I} x_j a_i^2 w_{ij}\\ &= a_i^2\left(t_i - \sum_{j\in I} x_j w_{ij}\right)\\ &= a_i^2 \cdot \tilde{b}_i(x)\end{aligned}$$

We used that $v_{ij} = v_{ji}$ and that $a_j a_i v_{ij} = w_{ij} a_i^2$.

Also, we have

$$\frac{\partial^2\phi^Q}{\partial x_i^2}(x) = -a_i^2 w_{ii} = -a_i^2 < 0.$$

Therefore, if $b_i(x) = \tilde{b}_i(x)$, then $\phi^Q(x)$ is maximal when $\tilde{b}_i(x) = b_i(x) = x_i$. If $\tilde{b}_i(x) < 0$ for some x_{-i}, then the derivative of ϕ^Q with respect to x_i is uniformly negative on $[0, \overline{x}_i]$, so the maximum is achieved when $x_i = 0$. On the other hand, when $\overline{x}_i < \tilde{b}_i(x)$, then the derivative of ϕ^Q with respect to x_i is positive on $[0, \overline{x}_i]$, and hence it takes its maximum in $\overline{x}_i$.

Therefore, for a fixed x_{-i} vector the function ϕ^Q is maximal when player i is in her best response, so ϕ^Q is a best-response potential function. ■

Proof of 3 ⇒ 2. Suppose that there exists a quadratic best-response potential function, ϕ^Q, for a game G on W. We show that the matrix W is sign symmetric and transitive.

Let the potential function be given as follows.

$$\phi^Q(x) = \sum_{i\in I} p_i x_i - \frac{1}{2}\sum_{i\in I} q_{ii} x_i^2 - \sum_{i,j\in I, i>j} q_{ij} x_i x_j,$$

where $p_i \in \mathbb{R}$ for every $i \in I$ and $q_{ij} \in \mathbb{R}$ for every $i, j \in I$, $i \geq j$. For $j > i$, we set $q_{ij} = q_{ji}$ for convenience. With this notation, the partial derivative of ϕ^Q is:

$$\frac{\partial\phi^Q}{\partial x_i}(x) = p_i - q_{ii}x_i - \sum_{j\in I\setminus\{i\}} q_{ij} x_j. \tag{8}$$

The function ϕ^Q is a best-response potential of the weighted network game G, so for every $x \in X$ and for every $i \in I$, the partial derivative of ϕ^Q is zero exactly when player i is in her best response. Note that we have $0 < t_i < \overline{x}_i$ for every $i \in I$, this means that $t_i = b_i(0) = \tilde{b}_i(0) \in (0, \overline{x}_i)$ for every $i \in I$. Therefore, there exists a neighborhood of 0 where each player's unconstrained best response is equal to her best response. Let $\varepsilon > 0$ be so that for every $x \in [0, \varepsilon]^n$ and for all $i \in I$ we have $b_i(x) = \tilde{b}_i(x)$. For a fixed i, the functions $\frac{\partial \phi^Q}{\partial x_i}(x)$ and $\tilde{b}_i(x) - x_i$ are both linear in x and they have the same zero set when $x \in [0, \varepsilon]^n$. Hence, they must be equal up to a constant factor: there exists $d_i \neq 0$ such that we have

$$t_i - \sum_{j \in I} w_{ij} x_j = \tilde{b}_i(x) - x_i = d_i \frac{\partial \phi^Q}{\partial x_i}(x) = d_i \left(p_i - \sum_{j \in I \setminus \{i\}} q_{ij} x_j - q_{ii} x_i \right) \tag{9}$$

Additionally, as we are maximizing ϕ^Q, the second derivative of ϕ^Q with respect to x_i has to be negative, so $q_{ii} > 0$ for every $i \in I$. From (9), we get the following for every $i, j \in I$.

$$\begin{aligned} t_i &= d_i \cdot p_i \\ w_{ii} = 1 &= d_i \cdot q_{ii} \\ w_{ij} &= d_i \cdot q_{ij} \end{aligned} \tag{10}$$

Since $q_{ii} > 0$, we must have $d_i > 0$ as well for all $i \in I$.

The first two equations give no constraints for W. We have to show that if (10) holds for all $i, j \in I$, that implies the transitivity and the sign-symmetry of the weight matrix W.

Since the d_i's are positive and $q_{ij} = q_{ji}$, we have

$$\operatorname{sgn}(w_{ij}) = \operatorname{sgn}(d_i q_{ij}) = \operatorname{sgn}(q_{ij}) = \operatorname{sgn}(q_{ji}) = \operatorname{sgn}(d_j q_{ji}) = \operatorname{sgn}(w_{ji}),$$

so the matrix W is sign-symmetric.

Now let $3 \leq s \leq n$, then for all $i_1, i_2, ..., i_s \in I$ pairwise distinct, we have

$$\begin{aligned} w_{i_1 i_2} w_{i_2 i_3} \dots w_{i_{s-1} i_s} w_{i_s i_1} &= (d_{i_1} q_{i_1 i_2})(d_{i_2} q_{i_2 i_3}) \dots (d_{i_{s-1}} q_{i_{s-1} i_s})(d_{i_s} q_{i_s i_1}) \\ &= (d_{i_1} d_{i_2} \dots d_{i_s})(q_{i_1 i_2} q_{i_2 i_3} \dots q_{i_{s-1} i_s} q_{i_s i_1}) \\ &= (d_{i_1} d_{i_2} \dots d_{i_s})(q_{i_2 i_1} q_{i_3 i_2} \dots q_{i_s i_{s-1}} q_{i_1 i_s}) \\ &= (d_{i_2} q_{i_2 i_1})(d_{i_3} q_{i_3 i_2}) \dots (d_{i_s} q_{i_s i_{s-1}})(d_{i_1} q_{i_1 i_s}) \\ &= w_{i_2 i_1} w_{i_3 i_2} \dots w_{i_s i_{s-1}} w_{i_1 i_s} \end{aligned}$$

using $w_{ij} = d_i q_{ij}$ for every $i, j \in I$. Therefore, the matrix W is transitive. ■

Proof of 2 ⇒ 1. Suppose that the matrix W is transitive and sign-symmetric. We would like to find a scaling vector $a \in \mathbb{R}^n$, $a > 0$, such that the rescaled matrix is symmetric, i.e., for every pair $i, j \in I$ we have $w_{ij}a_i/a_j = w_{ji}a_j/a_i$.

First, let us assume that the graph of W is connected, so there exists a path between any two players. If the graph is not connected, we follow the described algorithm for every connected component of the graph separately in order to define the vector a.

We start by ordering the players in the following way. Choose player 1 arbitrarily. Let the neighbors of player 1 be $2, \dots, n_1$. Players $n_1 + 1, \dots, n_2$ are those neighbors of 2 that are not neighbors of 1, and so on: players $n_k + 1, n_k + 2, \dots, n_{k+1}$ are those neighbors of player $k + 1$ that are not neighbors of players $1, 2, \dots, k$. (If there are no such players, then we have $n_k = n_{k+1}$.) Due to the sign-symmetry of W, if two players are neighbors, there is a directed edge between them in both directions. Hence, since W is connected, there exists a directed path from player 1 to every player, so after at most n steps we reach all players.

Let $a_1 = 1$. We define all a_j's for $j \geq 2$ recursively in the following way. Suppose we have already defined $a_1, a_2, ..., a_{j-1}$, and $n_{i-1} < j \leq n_i$, so player j is a neighbor of i but not of players $1, ..., i - 1$. In other words, i is the neighbor of j with the smallest index. Clearly $i < j$, so a_i is already defined. Also, we have $w_{ji} \neq 0 \neq w_{ij}$, since i and j are neighbors. Let

$$a_j = a_i \frac{\sqrt{|w_{ij}|}}{\sqrt{|w_{ji}|}}. \tag{11}$$

Now we show that with this scaling vector a, for any $k, \ell \in I$ we have $v_{k\ell} = v_{\ell k}$. Take any $k, \ell \in I$. If $k = \ell$, then $w_{k\ell} = v_{k\ell} = v_{\ell k} = w_{\ell k} = 1$, so we can assume that $k \neq \ell$. Let $i_1 \in I$ be so that k is a neighbor of i_1, but not of players $1, \dots, i_1 - 1$, so $n_{i_1 - 1} < k \leq n_{i_1}$. Similarly, let $j_1 \in I$ be so that $n_{j_1 - 1} < \ell \leq n_{j_1}$. For every $m \in \mathbb{N}$, we define i_m and j_m recursively:

$$i_{m+1} = \min\{i \in I : w_{i_m i} \neq 0\},$$
$$j_{m+1} = \min\{j \in I : w_{j_m j} \neq 0\}.$$

Both sequences are decreasing, and they both stabilize when they reach 1. Let $r, s \in \mathbb{N}$ be minimal so that $i_r = j_s$. Note that this is going to happen, the latest when they are both equal to 1, and also note that from this point on, the two sequences coincide. Therefore, we have $k > i_1 > \dots > i_r$ and $\ell > j_1 > \dots > j_s$, and the numbers $k, i_1, \dots, i_{r-1}, \ell, j_1, \dots, j_s$ are distinct.

We can calculate a_k as follows.

$$\begin{aligned}
a_k &= a_{i_1}\frac{\sqrt{|w_{i_1k}|}}{\sqrt{|w_{ki_1}|}}\\
&= a_{i_2}\frac{\sqrt{|w_{i_2i_1}|}}{\sqrt{|w_{i_1i_2}|}}\frac{\sqrt{|w_{i_1k}|}}{\sqrt{|w_{ki_1}|}}\\
&\vdots\\
&= a_{i_r}\frac{\sqrt{|w_{i_ri_{r-1}}|}}{\sqrt{|w_{i_{r-1}i_r}|}}\cdots\frac{\sqrt{|w_{i_2i_1}|}}{\sqrt{|w_{i_1i_2}|}}\frac{\sqrt{|w_{i_1k}|}}{\sqrt{|w_{ki_1}|}}\\
&= a_{i_r}\left|\frac{w_{i_ri_{r-1}}\dots w_{i_2i_1}w_{i_1k}}{w_{i_{r-1}i_r}\dots w_{i_1i_2}w_{ki_1}}\right|^{\frac{1}{2}}.
\end{aligned}$$

Similarly, we have

$$a_\ell = a_{j_s}\left|\frac{w_{j_sj_{s-1}}\dots w_{j_2j_1}w_{j_1\ell}}{w_{j_{s-1}j_s}\dots w_{j_1j_2}w_{\ell j_1}}\right|^{\frac{1}{2}}.$$

Now we can compute $v_{k\ell}$ using that $i_r = j_s$.

$$\begin{aligned}
v_{k\ell} &= \frac{a_k}{a_\ell}w_{k\ell} = a_k(a_\ell)^{-1}w_{k\ell}\\
&= \left(a_{i_r}\left|\frac{w_{i_ri_{r-1}}\dots w_{i_2i_1}w_{i_1k}}{w_{i_{r-1}i_r}\dots w_{i_1i_2}w_{ki_1}}\right|^{\frac{1}{2}}\right)\left(a_{j_s}\left|\frac{w_{j_sj_{s-1}}\dots w_{j_2j_1}w_{j_1\ell}}{w_{j_{s-1}j_s}\dots w_{j_1j_2}w_{\ell j_1}}\right|^{\frac{1}{2}}\right)^{-1} w_{k\ell}\\
&= \left|\frac{(w_{i_ri_{r-1}}\dots w_{i_2i_1}w_{i_1k})(w_{j_{s-1}j_s}\dots w_{j_1j_2}w_{\ell j_1})}{(w_{i_{r-1}i_r}\dots w_{i_1i_2}w_{ki_1})(w_{j_sj_{s-1}}\dots w_{j_2j_1}w_{j_1\ell})}\right|^{\frac{1}{2}} w_{k\ell}\\
&= \left|\frac{w_{i_1k}w_{i_2i_1}\dots w_{i_ri_{r-1}}w_{j_{s-1}j_s}\dots w_{j_1j_2}w_{\ell j_1}}{w_{ki_1}w_{i_1i_2}\dots w_{i_{r-1}i_r}w_{j_sj_{s-1}}\dots w_{j_2j_1}w_{j_1\ell}}\right|^{\frac{1}{2}}\left|\frac{w_{k\ell}}{w_{\ell k}}\right|^{\frac{1}{2}}\cdot \operatorname{sgn}(w_{k\ell})\sqrt{|w_{k\ell}w_{\ell k}|}\\
&= \left|\frac{w_{i_1k}w_{i_2i_1}\dots w_{j_si_{r-1}}w_{j_{s-1}j_s}\dots w_{j_1j_2}w_{\ell j_1}w_{k\ell}}{w_{ki_1}w_{i_1i_2}\dots w_{i_{r-1}j_s}w_{j_sj_{s-1}}\dots w_{j_2j_1}w_{j_1\ell}w_{\ell k}}\right|^{\frac{1}{2}} \operatorname{sgn}(w_{k\ell})\sqrt{|w_{k\ell}w_{\ell k}|}
\end{aligned}$$

By the transitivity assumption for $k, i_1, \dots i_{r-1}, j_s, \dots j_1, \ell \in I$, we have

$$w_{ki_1}w_{i_1i_2}\dots w_{i_{r-1}j_s}w_{j_sj_{s-1}}\dots w_{j_2j_1}w_{j_1\ell}w_{\ell k} = w_{i_1k}w_{i_2i_1}\dots w_{j_si_{r-1}}w_{j_{s-1}j_s}\dots w_{j_1j_2}w_{\ell j_1}w_{k\ell},$$

and hence $v_{k\ell} = \operatorname{sgn}(w_{k\ell})\sqrt{|w_{k\ell}w_{\ell k}|}$. Similarly, we have $v_{\ell k} = sgn(w_{\ell k})\sqrt{|w_{\ell k}w_{k\ell}|}$. Since W is sign-symmetric, these two numbers are equal, so $v_{k\ell} = v_{\ell k}$ for all $k, \ell \in I$.

Finally, we show that the statement holds even if the graph of W is not connected. In this case we order the players in each component and define the a_i's for the components separately as described in the beginning of the proof. Now take any $i, j \in I$. If i and j are the same connected component, we have already shown that $v_{ij} = v_{ji}$. If they are in different components, we know that $w_{ij} = w_{ji} = 0$, therefore $v_{ij} = 0 = v_{ji}$. This concludes the proof. ■

Proofs for Section 5

Let us introduce some notations and terminology that will be used in the proofs of this section.

Recall that a matrix norm $\|\cdot\|\colon \mathbb{C}^{m\times n} \to \mathbb{R}_+$ is an *induced norm* if it is induced by vector norms on $\mathbb{C}^m$ and $\mathbb{C}^n$, i.e., there exist norms $\|\cdot\|_{\mathbb{C}^m}\colon \mathbb{C}^m \to \mathbb{R}_+$ and $\|\cdot\|_{\mathbb{C}^n}\colon \mathbb{C}^n \to \mathbb{R}_+$ such that for $M \in \mathbb{C}^{m\times n}$ we have

$$\|M\| = \sup\{\|Mx\|_{\mathbb{C}^m} : x \in \mathbb{C}^n, \|x\|_{\mathbb{C}^n} = 1\}.$$

This definition implies that we have

$$\|Mx\|_{\mathbb{C}^m} \le \|M\| \cdot \|x\|_{\mathbb{C}^n} \tag{12}$$

for every $M \in \mathbb{C}^{m\times n}$ and $x \in \mathbb{C}^n$.

We will use a variation of the ∞-norm: the weighted maximum norm. Fix a weight vector $u \in \mathbb{R}^n$, $u > 0$. For $x \in \mathbb{C}^n$, let

$$\|x\|_\infty^u = \max\{|x_i|/u_i : 1 \le i \le n\}.$$

The induced matrix norm is the following: for $M = (m_{ij}) \in \mathbb{C}^{n\times n}$, we have

$$\|M\|_\infty^u = \sup\{\|Mx\|_\infty^u : \|x\|_\infty^u = 1\} = \max\left\{\frac{1}{u_i}\sum_{j=1}^n u_j|m_{ij}| : 1 \le i \le n\right\}. \tag{13}$$

For a vector $u \in \mathbb{R}^n$, $u > 0$, let $u^{-1} \in \mathbb{R}^n$ denote the vector with entries u_i^{-1}. Let $D_u = \mathrm{diag}(u_1, u_2, \ldots, u_n) \in \mathbb{R}^{n\times n}$, by (13), we have

$$\|M\|_\infty^u = \|D_u^{-1} M D_u\|_\infty.$$

If W is a network and $a \in \mathbb{R}^n$, $a > 0$ is a scaling vector, then for the rescaled matrix V we have $v_{ij} = w_{ij}a_i/a_j$. Equivalently, $V = D_a W D_a^{-1}$. Therefore, we have

$$\|W\|_\infty^{a^{-1}} = \|D_{a^{-1}}^{-1} W D_{a^{-1}}\|_\infty = \|D_a W D_a^{-1}\|_\infty = \|V\|_\infty. \tag{14}$$

For a matrix $M = (m_{ij})_{1\le i,j\le n} \in \mathbb{C}^{n\times n}$, we will denote by $|M| \in \mathbb{R}_+^{n\times n}$ the matrix $(|m_{ij}|)_{1\le i,j\le n}$. We will use the following statement.

Proposition A.1 (Perron-Frobenius Theorem)**.** *Let $M \in \mathbb{R}_+^{n\times n}$. Then, there exists a vector $z \ge 0$, $z \ne 0$ such that $Mz = \rho(M)z$.*

Furthermore, for any $\varepsilon > 0$ there exists a vector $u > 0$ such that $\rho(M) < \|M\|_\infty^u < \rho(M) + \varepsilon$.

For a proof see Chapter 2, Proposition 6.6 of Bertsekas and Tsitsiklis (1989).

Proposition 5.3

Now we can characterize the networks that can be rescaled into one with weak externalities.

Proof of Proposition 5.3. Let us start by proving the equivalence of *2*, *3*, and *4*:
4 $\Rightarrow$ *2*: Assume that $\rho(|W| - I_n) < 1$. Then, by Proposition A.1 for $0 < \varepsilon < 1 - \rho(|W| - I_n)$, there exists a vector $u \in \mathbb{R}^n$, $u > 0$ such that $\||W| - I_n\|_\infty^u < 1$. Let us use $u^{-1} = a$ as a scaling vector, i.e., let $v_{ij} = w_{ij} a_i / a_j$. Then, we have

$$1 > \||W| - I_n\|_\infty^u = \||W| - I_n\|_\infty^{a^{-1}} = \||V| - I_n\|_\infty = \|V - I_n\|_\infty,$$

so the matrix V is row diagonally dominant. In other words, V has weak influences.
2 $\Rightarrow$ *3*: Let $a \in \mathbb{R}^n$, $a > 0$ be a scaling vector so that the rescaled matrix V has weak influences, i.e., V is a row diagonally dominant matrix. Therefore, $\||V| - I_n\|_\infty = \|V - I_n\|_\infty < 1$, so by (14), we have

$$\||W| - I_n\|_\infty^{a^{-1}} = \|W - I_n\|_\infty^{a^{-1}} = \|V - I_n\|_\infty = \||V| - I_n\|_\infty < 1.$$

For any induced matrix norm $\|\cdot\|$, we have $\|MN\| \leq \|M\|\|N\|$ for any matrices M, N. Therefore, $\|M^k\| \leq \|M\|^k$ for any $k \in \mathbb{N}$ and any matrix M. Hence,

$$\lim_{k\to\infty} \|(|W| - I_n)^k\|_\infty^{a^{-1}} \leq \lim_{k\to\infty} \left(\||W| - I_n\|_\infty^{a^{-1}}\right)^k = 0,$$

since $\||W| - I_n\|_\infty^{a^{-1}} < 1$. The norm of $(|W| - I_n)^k$ converges to 0, this is only possible if the matrices converge to the 0 matrix, so we have $\lim_{k\to\infty}(|W| - I_n)^k = 0$.
3 $\Rightarrow$ *4*: Assume that the limit is 0. Let λ be any eigenvalue of $|W| - I_n$, and $z \neq 0$ the corresponding eigenvector. Note that z is also an eigenvector of $(|W| - I_n)^k$ with eigenvalue λ^k. We have

$$0 = \left(\lim_{k\to\infty} (|W| - I_n)^k\right) z = \lim_{k\to\infty} (|W| - I_n)^k z = \lim_{k\to\infty} \lambda^k z = \left(\lim_{k\to\infty} \lambda^k\right) z.$$

Since $z \neq 0$, we must have $\lim_{k\to\infty} \lambda^k = 0$, hence $|\lambda| < 1$. This is true for any eigenvalue, so $\rho(|W| - I_n) < 1$.

Hence, conditions *2*, *3*, and *4* are equivalent. Now notice that for any matrix M, we have $\lim_{k\to\infty} M^k = 0$ if and only if $\lim_{k\to\infty}(M^\top)^k = 0$. Therefore, we have the following equivalences: the network W can be rescaled into a row diagonally dominant matrix $\Leftrightarrow \lim_{k\to\infty}(|W| - I_n)^k = 0$ $\Leftrightarrow \lim_{k\to\infty}((|W| - I_n)^\top)^k = 0 \Leftrightarrow W^\top$ can be rescaled into a row diagonally dominant matrix $\Leftrightarrow$ W can be rescaled into a column diagonally dominant matrix.

Column diagonal dominance means exactly that the network has weak externalities, so we proved the equivalence of *1* with the other three statements. ■

Proposition 5.4

Lemma A.2. *Recall that* $b\colon X \to X$ *denotes the best response mapping. For any vector* $u \in \mathbb{R}^n$, $u > 0$ *and for all* $x, x' \in X$, *we have*

$$\|b(x) - b(x')\|_\infty^u \leq \|W - I_n\|_\infty^u \|x - x'\|_\infty^u.$$

Proof. First, consider the unconstrained best responses $\tilde{b}(x)$ and $\tilde{b}(x')$. We have

$$\begin{aligned}
\|\tilde{b}(x) - \tilde{b}(x')\|_\infty^u &= \|(t - (W - I_n)x) - (t - (W - I_n)x')\|_\infty^u \\
&= \|(W - I_n)(x' - x)\|_\infty^u \\
&\leq \|W - I_n\|_\infty^u \|x - x'\|_\infty^u \qquad \text{by (12).}
\end{aligned}$$

Notice that for every $i \in I$, we have $|b_i(x) - b_i(x')| \leq |\tilde{b}_i(x) - \tilde{b}_i(x')|$. Indeed, without loss of generality, we can assume that $\tilde{b}_i(x) \leq \tilde{b}_i(x')$ and we can verify the inequality in all cases:

- If $\tilde{b}_i(x) \leq \tilde{b}_i(x') \leq 0$, then $b_i(x) = b_i(x') = 0$, so $|b_i(x) - b_i(x')| = 0 \leq |\tilde{b}_i(x) - \tilde{b}_i(x')|$.
- If $\overline{x}_i \leq \tilde{b}_i(x) \leq \tilde{b}_i(x')$, then $b_i(x) = b_i(x') = \overline{x}_i$, so $|b_i(x) - b_i(x')| = 0 \leq |\tilde{b}_i(x) - \tilde{b}_i(x')|$.
- If $\tilde{b}_i(x) \leq 0 \leq \tilde{b}_i(x')$ or if $\tilde{b}_i(x) \leq \overline{x}_i \leq \tilde{b}_i(x')$, then $\tilde{b}_i(x) \leq b_i(x) \leq b_i(x') \leq \tilde{b}_i(x')$, and hence $|b_i(x) - b_i(x')| \leq |\tilde{b}_i(x) - \tilde{b}_i(x')|$.
- If $0 \leq \tilde{b}_i(x) \leq \tilde{b}_i(x') \leq \overline{x}_i$, then $b_i(x) = \tilde{b}_i(x)$ and $b_i(x') = \tilde{b}_i(x')$, so $|b_i(x) - b_i(x')| = |\tilde{b}_i(x) - \tilde{b}_i(x')|$.

Therefore, we have

$$\begin{aligned}
\|b_i(x) - b_i(x')\|_\infty^u &= \sum_{i \in I} |b_i(x) - b_i(x')|/u_i \\
&\leq \sum_{i \in I} |\tilde{b}_i(x) - \tilde{b}_i(x')|/u_i \\
&= \|\tilde{b}(x) - \tilde{b}(x')\|_\infty^u \\
&\leq \|W - I_n\|_\infty^u \|x - x'\|_\infty^u.
\end{aligned}$$

This concludes the proof of the lemma. ■

Proof of Proposition 5.4. By Proposition 5.3, there exists a scaling vector $a \in \mathbb{R}^n$, $a > 0$ such that the rescaled matrix is row diagonally dominant, i.e., we have $\|W - I_n\|_\infty^{a^{-1}} = \|V - I_n\|_\infty < 1$.

Assume that x^* and x^{**} are two different Nash equilibria, i.e., we have $b(x^*) = x^*$, $b(x^{**}) = x^{**}$ and $x^* \neq x^{**}$. By the assumption that $\|W - I_n\|_\infty^{a^{-1}} < 1$ and by Lemma A.2, we get

$$\|x^* - x^{**}\|_\infty^{a^{-1}} = \|b(x^*) - b(x^{**})\|_\infty^{a^{-1}} \leq \|W - I_n\|_\infty^{a^{-1}} \|x^* - x^{**}\|_\infty^{a^{-1}} < \|x^* - x^{**}\|_\infty^{a^{-1}},$$

which is a contradiction. Hence, there exists a unique Nash equilibrium. ■

Proposition 5.5

Lemma A.3. *If the network W can be rescaled into a network with weak externalities using the vector $a \in \mathbb{R}^n$, $a > 0$, then for every $j \in I$, we have $\sum_{i\in I\setminus\{j\}} a_i|w_{ij}| < a_j$.*

Proof. Take a vector $a \in \mathbb{R}^n$, $a > 0$ such that the rescaled network V is with weak externalities. By definition, this means that for every $j \in I$, we have

$$\sum_{i\in I\setminus\{j\}} |v_{ij}| < |v_{jj}| = 1$$
$$\sum_{i\in I\setminus\{j\}} \frac{a_i}{a_j}|w_{ij}| < |w_{jj}| = 1$$
$$\sum_{i\in I\setminus\{j\}} a_i|w_{ij}| < a_j.$$

■

Lemma A.4. *Let $i, j \in I$ and let $(x^k)_{k\in\mathbb{N}}$ be a best-response dynamic. If player i's action changes by Δ, player j's best response changes by a maximum of $|\Delta \cdot w_{ji}|$, i.e., we have $|b_j(x^k) - b_j(x^{k+1})| \leq |w_{ji}| \cdot |x_i^k - x_i^{k+1}|$ for any $k \in \mathbb{N}$.*

Proof. By (3), the unconstrained best response function of player j is $\tilde{b}_j(x) = t_j - \sum_{i\in I\setminus\{j\}} w_{ji}x_i$. Therefore, if $\tilde{b}_j(x) \in (0, \overline{x}_j)$, then we have $\tilde{b}_j(x) = b_j(x)$, so $\partial b_j(x)/\partial x_i = w_{ji}$. If $\tilde{b}_j(x) \notin [0, \overline{x}_j]$, then we know that $b_j(x) \in \{0, \overline{x}_j\}$, so $\partial b_j(x)/\partial x_i = 0$. Hence, by the mean value theorem, if player j's action changes by Δ, her best response can change by at most $|\Delta \cdot w_{ji}|$. ■

Proof of Proposition 5.5. Let $i \in I$ and fix $x \in X$. We need to prove that only $b_i(x)$ maximizes $\phi'(\cdot, x_{-i})$. Assume that we have $x_i^1 \in \operatorname{argmax}_{x_i\in X_i} \phi'(x_i, x_{-i})$. Let $x^1 = (x_i^1, x_{-i})$ and $i^1 = i$. Then $x^2 = (b_i(x^1), x_{-i}) = (x_j^2)_{j\in I}$. We have that $\phi'(x^1) \geq \phi'(x^2)$ since $x_i^1 \in \operatorname{argmax}_{x_i\in X_i} \phi'(x_i, x_{-i}^1)$.

Therefore, we have

$$
\begin{aligned}
0 &\geq \phi'(x^2) - \phi'(x^1) \\
&= -\left(\sum_{j\in I\setminus\{i\}} a_j|x_j^2 - b_j(x^2)|\right) - \left(-\sum_{j\in I} a_j|x_j^1 - b_j(x^1)|\right) \\
&= a_i|x_i^1 - b_i(x^1)| + \sum_{j\in I\setminus\{i\}} a_j\left(|x_j^1 - b_j(x^1)| - |x_j^1 - b_j(x^2)|\right) \\
&\geq a_i|x_i^1 - b_i(x^1)| - \sum_{j\in I\setminus\{i\}} a_j|b_j(x^1) - b_j(x^2)| \\
&\geq a_i|x_i^1 - b_i(x^1)| - \sum_{j\in I\setminus\{i\}} a_j|w_{ji}||x_i^1 - x_i^2| \qquad \text{by Lemma A.4} \\
&= \left(a_i - \sum_{j\in I\setminus\{i\}} a_j|w_{ji}|\right)|x_i^1 - b_i(x^1)| \\
&\geq 0 \qquad \text{by Lemma A.3.}
\end{aligned}
$$

Hence, we must have equality everywhere. Since $a_i - \sum_{j\in I\setminus\{i\}} a_j|w_{ji}| > 0$, equality holds in the last line if and only if $|x_i^1 - b_i(x^1)| = 0$, i.e., iff $x_i^1 = b_i(x^1) = b_i(x)$. Thus, we have $\operatorname{argmax}_{x_i\in X_i} \phi'(x_i, x_{-i}) = \{b_i(x)\}$, so ϕ' is a best-response potential function. ■

Theorem 5.6

Proof of Theorem 5.6. By Proposition 5.3, there exists a scaling vector $a \in \mathbb{R}^n$, $a > 0$ such that the rescaled matrix is row diagonally dominant, i.e., we have $\|W - I_n\|_\infty^{a^{-1}} = \|V - I_n\|_\infty < 1$. Let $\gamma = \|W - I_n\|_\infty^{a^{-1}}$, then $0 \leq \gamma < 1$.

By Proposition 5.4, there is a unique Nash equilibrium, let us denote it by x^*. Notice that a BRD is a BRAD with parameter 0, so it is enough to prove the statement for BRAD's. Consider any BRAD $(x^k)_{k\in\mathbb{N}}$ with approach parameter $0 \leq \beta < 1$.

By the definition of the weighted maximum norm, for every $x, y \in X$ we have

$$
a_i|x_i - y_i| \leq \|x - y\|_\infty^{a^{-1}}. \tag{15}
$$

Hence, for an arbitrary $k \in \mathbb{N}$, we have

$$
\begin{aligned}
a_{i^k}|b_{i^k}(x^k) - x_{i^k}^*| &= a_{i^k}|b_{i^k}(x^k) - b_{i^k}(x^*)| \\
&\leq \|b(x^k) - b(x^*)\|_\infty^{a^{-1}} \qquad \text{by (15)} \\
&\leq \|W - I_n\|_\infty^{a^{-1}}\|x^k - x^*\|_\infty^{a^{-1}} \qquad \text{by Lemma A.2} \\
&= \gamma \cdot \|x^k - x^*\|_\infty^{a^{-1}}.
\end{aligned} \tag{16}
$$

For the next claim, note that since $\gamma+\beta-\gamma\beta = 1-(1-\gamma)(1-\beta)$, we have $0 \le \gamma+\beta-\gamma\beta < 1$.

Claim A.5. *For every $k \in \mathbb{N}$, we have*

$$a_{i^k}|x_{i^k}^{k+1} - x_{i^k}^*| \le (\gamma+\beta-\gamma\beta)\|x^k - x^*\|_\infty^{a^{-1}}.$$

Proof. Let us use the notation $D = a_{i^k}^{-1}\|x^k - x^*\|_\infty^{a^{-1}} \in \mathbb{R}_+$. By (15), we have $|x_{i^k} - x_{i^k}^*| \le a_{i^k}^{-1}\|x^k - x^*\|_\infty^{a^{-1}} = D$, and hence $x_{i^k}^k$ is contained in the interval of lenght $2D$ with midpoint $x_{i^k}^*$. By (16), $b_{i^k}(x^k)$ is in the interval of length $\gamma 2D$ centered at $x_{i^k}^*$.

For $p, q \in \mathbb{R}$, let us use the notation $[p,q] = [\min\{p,q\},\ \max\{p,q\}]$ for the interval between p and q.

From the definition of BRAD, we have $x_{i^k}^{k+1} \in [b_{i^k}(x^k),\ (1-\beta)b_{i^k}(x^k) + \beta x_{i^k}^k]$, since this is the β-contracted image of $[b_{i^k}(x^k),\ x_{i^k}^k]$ towards $b_{i^k}(x^k)$. This implies that $x_{i^k}^{k+1}$ is contained in the β-contracted image of $[x_{i^k}^* - D,\ x_{i^k}^* + D]$ around $b_{i^k}(x^k)$, which is a point of $[x_{i^k}^* - \gamma D,\ x_{i^k}^* + \gamma D]$. Hence, we get the worst upper bound for $x_{i^k}^{k+1}$ if $b_{i^k}(x^k)$ takes the maximal value in $[x_{i^k}^* - \gamma D,\ x_{i^k}^* + \gamma D]$, and the worst lower bound if $b_{i^k}(x^k)$ takes the minimal value in the interval. We can compute these bounds: if $b_{i^k}(x^k) = x_{i^k}^* + \gamma D$, then the contracted image of $x_{i^k}^* + D$ is

$$(1-\beta)(x_{i^k}^* + \gamma D) + \beta(x_{i^k}^* + D) = x_{i^k}^* + (\gamma+\beta-\gamma\beta)D.$$

Similarly, for $b_{i^k}(x^k) = x_{i^k}^* - \gamma D$ we get the lower bound

$$(1-\beta)(x_{i^k}^* - \gamma D) + \beta(x_{i^k}^* - D) = x_{i^k}^* - (\gamma+\beta-\gamma\beta)D.$$

Therefore, $x_{i^k}^{k+1} \in [x_{i^k}^* - (\gamma+\beta-\gamma\beta)D,\ x_{i^k}^* + (\gamma+\beta-\gamma\beta)D]$, and hence

$$a_{i^k}|x_{i^k}^{k+1} - x_{i^k}^*| \le (\gamma+\beta-\gamma\beta)\|x^k - x^*\|_\infty^{a^{-1}},$$

as desired. ∎

Claim A.6. *For every $m \in \mathbb{N}$ there exists $K(m) \in \mathbb{N}$ such that for all $k \ge K(m)$ we have*

$$\|x^k - x^*\|_\infty^{a^{-1}} \le (\gamma+\beta-\gamma\beta)^m \|x^0 - x^*\|_\infty^{a^{-1}}.$$

Proof. We prove the statement by induction on m. For $m = 0$, it clearly holds with $K(0) = 0$. Assume that $K(m-1)$ exists, and we would like to find $K(m)$.

Take an arbitrary $k \ge K(m-1)$ and let player i be the one who moves at time k. Then, we have

$$\begin{aligned}
a_i|x_i^{k+1} - x_i^*| &\le (\gamma+\beta-\gamma\beta)\|x^k - x^*\|_\infty^{a^{-1}} && \text{by Claim A.5} \\
&\le (\gamma+\beta-\gamma\beta)(\gamma+\beta-\gamma\beta)^{m-1}\|x^0 - x^*\|_\infty^{a^{-1}} && \text{by the induction hypothesis} \\
&= (\gamma+\beta-\gamma\beta)^m\|x^0 - x^*\|_\infty^{a^{-1}}.
\end{aligned}$$

Therefore, we can see that $a_i|x_i^\ell - x_i^*| \leq \gamma^m \|x^0 - x^*\|_\infty^{a^{-1}}$ for all $\ell > k$, since it is true after every move of player i, and it remains true in all other players' turns because that does not change the action of player i.

For every $i \in I$, let k_i be the first time player i moves after $K(m-1)$. Let

$$K(m) = \max\{k_i : i \in I\} + 1.$$

By time $K(m)$, every player moved at least once since $K(m-1)$, so for every $i \in I$ and all $k \geq K(m)$, we have $a_i|x_i^k - x_i^*| \leq (\gamma + \beta - \gamma\beta)^m \|x^0 - x^*\|_\infty^{a^{-1}}$. Therefore, we also have

$$\|x^k - x^*\|_\infty^{a^{-1}} = \max\{a_i|x_i^k - x_i^*| : i \in I\} \leq (\gamma + \beta - \gamma\beta)^m \|x^0 - x^*\|_\infty^{a^{-1}}.$$

This proves the statement for every $m \in \mathbb{N}$. ■

Now we can show the convergence of the BRAD $(x^k)_{k\in\mathbb{N}}$ to the Nash equilibrium x^*. Take any $\varepsilon > 0$, then there exists $m \in \mathbb{N}$ such that $(\gamma+\beta-\gamma\beta)^m \|x^0 - x^*\|_\infty^{a^{-1}} < \varepsilon$, since $\gamma+\beta-\gamma\beta < 1$. Therefore, if $k \geq K(m)$ from Claim A.6, then we have

$$\|x^k - x^*\|_\infty^{a^{-1}} \leq (\gamma + \beta - \gamma\beta)^m \|x^0 - x^*\|_\infty^{a^{-1}} < \varepsilon.$$

Hence, $(x^k)_{k\in\mathbb{N}}$ converges to the Nash equilibrium x^*. ■

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